A Framework for Intergration of Loss Function and Utility Function

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Risk preference plays an important role in the development of decision theory, it captures the individual's tendency or behavior in the presence of uncertainty. The utility function is the most commonly used tool to quantify the risk attitude, and the rational decision always associated with the maximum of expected utility. In addition, the loss function is extensively used in optimization and parameter estimation, and the best estimate is always seeking to minimize expected loss, which leads people to make a decision with a minimum error. However, utility function and loss function are often employed in different scenarios, which bring the difficulty to integrate these two frameworks. This research aims to investigate the relationship between utility function and loss function in different backgrounds with the commonly utilized exponential form and quadratic form. It ultimately establishes a workflow for the calibration of risk position in utility function and symmetry of the loss function, which provides a new perspective for understanding the loss functions and utility function in decision making.

Introduction

In the past few decades, utility function has had a profound impact on decision theory. It is an essential tool to measure the satisfaction of decision-makers and reflect people's preferences under risk (Eidsvik, Mukerji, & Bhattacharjya, 2015; Zou, Scholer, & Higgins, 2020). In addition, the framework of expected utility theory was first proposed by Dianiel Bernoulli and systematically organized by Von Neumann and Morgenstern in 1947 (Von Neumann, Morgenstern, & Kuhn, 2007). This theorem indicates that the rational decision is always associated with the maximum of expected utility. The decision-making problems in the petroleum industry, such as well placement optimization, could be expressed as follow (Vizcaino, 2019):

$$\underset{n}{\operatorname{argmax}}\, E\{U(X(n,m)\} = \underset{n}{\operatorname{argmax}} \int f(X(n,m))U(X(n,m))dX \quad (1)$$

Where $U(\cdot)$ and $X(\cdot)$ are the utility function and reward function, respectively. $f(\cdot)$ denotes the corresponding probability density function. n is a set of alternatives for the well position and m represents the number of realization in the stochastic simulation.

The other crucial concept in decision theory is the loss function, which is extensively utilized in optimization and parameter estimation (Chakraborty & Das, 2018; Meena, Arshad, & Gangopadhyay, 2018). It leads to an efficient decision by maximizing the estimates from the minimum of the expected loss. In the reservoir management, the random variable $Z(n,m)$ is often the petrophysical properties like porosity and permeability. The decision problem with loss function could be formulated as:

$$\underset{n}{\operatorname{argmax}}(\underset{Z^\star}{\operatorname{argmin}}\, E\{L(Z(n,m), Z^\star(n))\}) = \underset{n}{\operatorname{argmax}}(\underset{Z^\star}{\operatorname{argmin}} \int f(Z(n,m))L(Z(n,m), Z^\star(n))dZ) \quad (2)$$

Where $L(\cdot)$ is the loss function and $Z^\star(\cdot)$ is the estimate function. The desired decision is calculated from minimizing the expected loss on each position and then maximizing the estimate for all positions, which indicates two-steps optimization in the decision making with loss function under the resource management.

The utility function is usually available from questionnaires or interviews of decision-makers by 50-50 game or certainty equivalent (Guyaguler, Horne, et al., 2004; Walls, 2005). An exponential utility function is proposed for the simplified utility theory, which is commonly used in the industry to incorporate the risk preference. This utility function is extensively used in the monetary background, such as finance or economics, which has the characteristics of monotonic increasing and marginal utility decreasing. Berger (2013) defined the loss function is the negative of a utility function, this is evident in the same background. However, loss function is commonly used in the parameter estimation and utility function is widely utilized in the monetary related background, which caused the difficulty of integrating them in different backgrounds.

The purpose of this research is to investigate the relationship between utility and loss function. More specifically, the link between risk position in the utility function and symmetry of the loss function. An assumption of a rational decision-maker is made for the seeking of maximum expected utility, and all the decisions are uniformly distributed for simplicity. In addition, The concepts of transitional loss and transitional utility are utilized to establish this relationship. Although there is no determined relationship between them as the various function forms, the constructed framework still exhibits the possible potential between them.

Selecting a loss function and utility function

Many different forms of loss functions have been proposed so far, such as linear loss function, quadratic loss function or linex loss function and so on (Chen, 2019; Kinyanjui, Korir, et al., 2020). The quadratic loss function is employed in our study, it is a commonly used loss function because its form is similar to the mean square error in regression. Considering a non-negative random variable Z and its estimation $z^\star$. The quadratic loss function can be expressed:

$$L(z - z^\star) = \begin{cases} \lambda_2 (z - z^\star)^2 & z \geq z^\star \\ \lambda_1 (z - z^\star)^2 & z^\star > z \geq 0 \end{cases} \tag{3}$$

Where $z - z^\star$ is the error between the observed value z and the estimate $z^\star$. The $z - z^\star < 0$ means the overestimation, while $z - z^\star > 0$ stands for the underestimation. Moreover, the symmetry of loss function depends on whether the weights λ_1 and λ_2 are equal or not. The symmetric loss function has the same penalty for the loss and reward with $\lambda_1 = \lambda_2$. Similarly, the different penalty could be seen in asymmetric loss function when $\lambda_1 \neq \lambda_2$. The symmetry coefficient $\lambda = \frac{\lambda_2}{\lambda_1}$ is also called the loss-scale in this note, it could reflect the symmetric of the loss function. The quadratic loss function is categorized as a right-asymmetric loss function ($\lambda < 1$) , symmetric loss function ($\lambda = 1$) and left-asymmetric loss function ($\lambda > 1$) based on the penalty on the overestimation and underestimation (Figure 1).

Utility function also has various kinds of forms, for example, linear utility function, exponential utility function, power utility function and so forth (Gerber & Pafum, 1998; Niromandfam, Yazdankhah, & Kazemzadeh, 2020). The exponential utility function $U(x)$ is utilized in our research, and is of the form in Equation 4. The exponential form is suitable for the risk representation in the financial background (Cozzolino et al., 1977).

$$U(x) = \begin{cases} (1 - e^{-rx})/r & r \neq 0 \\ x & r = 0 \end{cases} \tag{4}$$

Where r is the risk tolerance coefficient, also named as the utility-scale, it describes the risk position of the firm under uncertainty. The $r > 0$ means the risk aversion, the $r = 0$ represents for the risk neutral, and the $r < 0$ implies for the risk acceptance (Figure 1). The x is the monetary value, it could be simply treated as the profit when $x > 0$ and loss when $x < 0$.

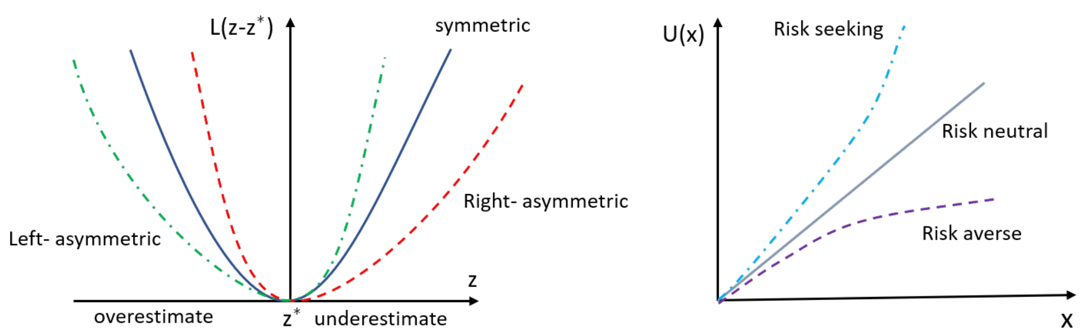

Figure 1: A sketch for the quadratic loss function (Left) and exponential utility function (Right)

According to Appendix 1, the optimal estimate is equal to the expected return in quadratic loss function when $\lambda_1 = \lambda_2$, and the expected utility is also equal to the expected return in exponential utility function when $r = 0$. This property indicates the equivalence between the symmetric quadratic loss function and exponential utility function with a risk-neutral position. In addition, another simplified utility function with power form, $U(x) = x^r$, would generate invalid values in the transitional loss-scale, which will be described in the latter section.

Optimal expected loss

The expected loss function is also known as risk function, it is an important index to measure the fitness of the estimate in the presence of uncertainty. Through taking the derivative of expected loss with respect to the estimate, the optimal estimate is the prediction value making the derivative result zero to minimize the expected loss. As for the decision with a uniform distribution within the interval between a and b, the optimal solution $z^\star$ is expressed in Equation 5, and its detailed steps in Appendix 2.

$$z^\star = \frac{a+\sqrt{\lambda}b}{1+\sqrt{\lambda}} \quad \text{or} \quad \frac{a-\sqrt{\lambda}b}{1-\sqrt{\lambda}} \quad \text{when} \quad \lambda \neq 1 \tag{5}$$

Since the optimal estimate $z^\star$ must be within the interval $[a,b]$ to be valid, two aspects are discussed according to the relative value of λ_1 and λ_2 in the asymmetric loss function (Figure 2).

① The $\lambda_1 > \lambda_2$ (or $\lambda < 1$) represents $\frac{a-\sqrt{\lambda}b}{1-\sqrt{\lambda}}$ is less than a, and $z^\star = \frac{a+\sqrt{\lambda}b}{1+\sqrt{\lambda}}$ is the optimal estimate.

② The $\lambda_1 < \lambda_2$ (or $\lambda > 1$) means $\frac{a-\sqrt{\lambda}b}{1-\sqrt{\lambda}}$ is greater than b, and $z^\star = \frac{a+\sqrt{\lambda}b}{1+\sqrt{\lambda}}$ is the optimal estimate.

In summary, the optimal estimate for the decision with a uniform distribution between a and b in the quadratic loss function is:

$$z^\star = \frac{a+\sqrt{\lambda}b}{1+\sqrt{\lambda}} \tag{6}$$

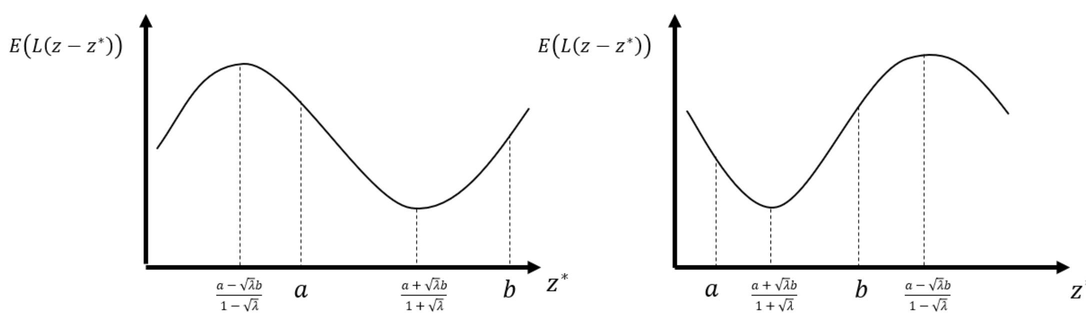

Figure 2: A sketch for the distribution of the optimal solution when $\lambda_1 \neq \lambda_2$. Left is for $\lambda_1 < \lambda_2$, and right is for $\lambda_1 > \lambda_2$, which shows the single valid solution in Equation 5

The transitional loss-scale λ_T refers to the loss-scale when the optimal estimates in different decisions are the same. Assuming the decision one has a uniform distribution in the interval between a and b, the optimal estimate for decision one is $z_1^\star = \frac{a+\sqrt{\lambda}b}{1+\sqrt{\lambda}}$. Similarly, the optimal estimate for decision two, uniform distribution with the interval of $[c,d]$, is expressed as $z_2^\star = \frac{c+\sqrt{\lambda}d}{1+\sqrt{\lambda}}$. The transitional loss-scale λ_T is shown in Equation 7 when $z_1^\star = z_2^\star$.

$$\lambda_T = (\frac{a-c}{d-b})^2 \tag{7}$$

In addition, let $g(\lambda) = (\frac{a-c}{d-b})^2 - \lambda$, the numerical solution λ_T is reached when $g(\lambda_T) = 0$.

Optimal expected utility

The exponential expected utility for a decision with a uniform distribution of interval between a and b is expressed in Equation 8.

$$\begin{aligned} E(U) &= \int_a^b \frac{1-e^{-rx}}{r}\frac{1}{b-a}dx \\ &= \frac{1}{r} + \frac{e^{-br}-e^{-ar}}{r^2(b-a)} \quad r \neq 0 \end{aligned} \tag{8}$$

Assuming two decisions are uniformly distributed in the intervals of $[a, b]$ and $[c, d]$, respectively. The transitional utility-scale r_T is the value when the expected utility of decision one $E(U_1)$ is equal to it is in decision two $E(U_2)$. However, the difference between $E(U_1)$ and $E(U_2)$ is very small in the exponential expected utility when the r becomes large, which brings the difficulty to compare them by subtraction.

$$E(U_1) - E(U_2) = \frac{1}{r} + \frac{e^{-br} - e^{-ar}}{r^2(b-a)} - \frac{1}{r} + \frac{e^{-dr} - e^{-cr}}{r^2(d-c)} = \frac{e^{-br} - e^{-ar}}{r^2(b-a)} - \frac{e^{-dr} - e^{-cr}}{r^2(d-c)} \tag{9}$$

An example of $r = 17$ is illustrated in Equation 10, which consists of two decisions with the intervals of $[2, 7]$ and $[3, 5]$, respectively. The result implies $E(U_1) - E(U_2)$ is approximately zero, it would cause the wrong numerical solution due to the limitation of precision in the program.

$$\begin{aligned} E(U_1)) =& \frac{1}{17} + \frac{e^{-7*17} - e^{-2*17}}{17^2 * (7-2)} = \frac{1}{17} - 1.186 * 10^{-18} \\ E(U_2)) =& \frac{1}{17} + \frac{e^{-5*17} - e^{-3*17}}{17^2 * (5-3)} = \frac{1}{17} - 1.228 * 10^{-25} \end{aligned} \tag{10}$$

Therefore, the comparison between $E(U_1)$ and $E(U_2)$ through the subtraction in Equation 9 is not recommended. A better way is to compare the quotient of $\frac{e^{-br}-e^{-ar}}{r^2(b-a)}$ and $\frac{e^{-dr}-e^{-cr}}{r^2(d-c)}$ with 1 in Equation 11, which is used to calculate the transitional utility-scale. In addition, the $f(r_T) = 0$ means the expected utility of decision one is the same with it is in decision two. The $f(r_T) = 0$ will generate two numerical solutions, the solution of zero, $\lim_{r \to 0} f(r) = 0$, is not valid as the assumption of $r \neq 0$ in Equation 8.

$$f(r) = 1 - \{\frac{e^{-br} - e^{-ar}}{r^2(b-a)} / \frac{e^{-dr} - e^{-cr}}{r^2(d-c)}\} = 1 - \frac{(e^{-br} - e^{-ar})(d-c)}{(e^{-dr} - e^{-cr})(b-a)} \tag{11}$$

The other simplified utility function is $\{U(x) = x^r; r > 0\}$. If the same steps are performed with the power utility function, two solutions could be solved for the transitional utility-scale: one solution is zero, the other solution is in the negative part. However, r_T should be a positive value in this utility form. Therefore, $\{U(x) = x^r; r > 0\}$ does not have a valid solution for transitional utility-scale, and the exponential utility function will be used in our research.

Link between transitional scale r_T and λ_T

Considering the decisions are uniformly distributed with different variances. The decision one has a distribution in the interval $[a, b]$, and decision two has a distribution with a higher variance in the interval $[c, d]$. An experiment, denoted as $[c, a, b, d]$, contains only these two decisions. Moreover, there are three positions for the distributions (Figure 3), the stochastic dominance theory indicates the distribution with high mean is always preferred in the separated distributions (Levy, 2015). Thus, the overlapped distributions is used to link these transitional scales.

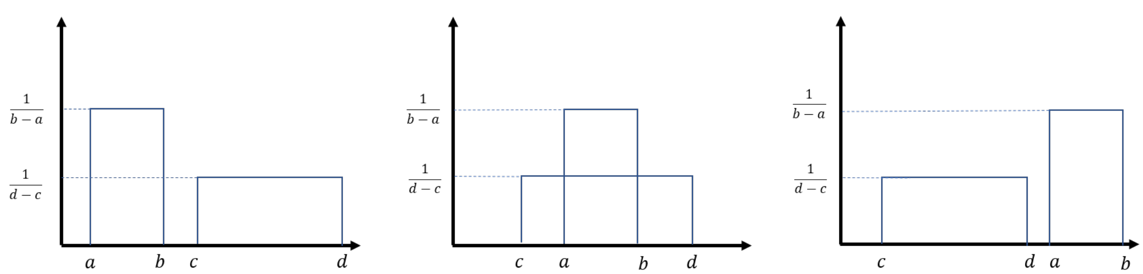

Figure 3: The position of two decisions with uniform distributions, left and right are separately distributed and the middle is the overlapped distributions

Both of the transitional loss-scale λ_T and transitional utility-scale r_T are able to reflect the balance of two different decisions. More specifically, they could lead to the same final decision under a set of choices, which implies the equivalence between them under this state. Therefore, the investigation of the relationship between loss function and utility function by comparing these transitional scales. The functions in Equation 12 summaries from Equation 7 and Equation 11 for calculating the numerical solution.

$$g(\lambda) = (\frac{a-c}{d-b})^2 - \lambda \text{ and } f(r) = 1 - \frac{(e^{-br} - e^{-ar})(d-c)}{(e^{-dr} - e^{-cr})(b-a)} \quad (12)$$

The transitional loss-scale λ_T and transitional utility-scale r_T are solved from $g(\lambda_T) = 0$ and $f(r_T) = 0$, they are the intersections of $g(\lambda)$ and $f(r)$ with x-axis. A one-experimental example is illustrated in Figure 3 with the uniform distributions of $[2, 7]$ and $[3, 5]$, respectively. The result indicates $f(r)$ only has one intersection with the x-axis, while the $g(\lambda)$ has two intersections with the x-axis. Therefore, the transitional loss-scale is 0.24, and the effective transitional utility-scale is 0.615. Through the utility function has the other intersection of zero, but that is invalid.

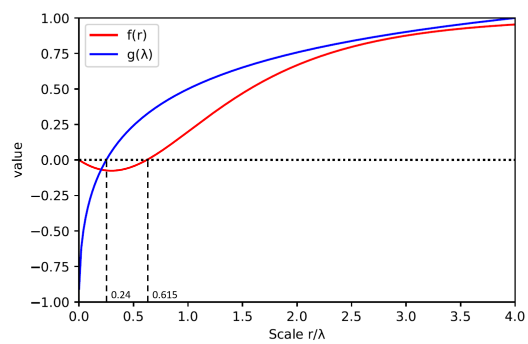

Figure 4: An example of numerical solutions for the transitional scales

In order to establish a comprehensive relationship, 200 groups of experiments are conducted by changing the distributions' intervals for all decisions. Since the distributions in the overlapped decisions usually across a limited range, their intervals are determined by four points $[c, a, b, d]$. These points are sequentially generated: Firstly, randomly sample the point c within the interval of [0.1,1]. Secondly, the spaces between every two adjacent points are randomly generated in the interval of $[0.1, 1]$, and lastly, the distributions are obtained by the accumulative sum of these random increments. 200 pairs of transitional loss-scale λ_T and corresponding transitional utility-scale r_T are recorded to establish the cross plot in Figure 5. Moreover, the original scatter plot shows an exponential-similar trend with a based less than 1. Therefore, the transitional loss-scale λ_T is also transformed with log-scale $ln\lambda_T$ for an intuitive display of its large range.

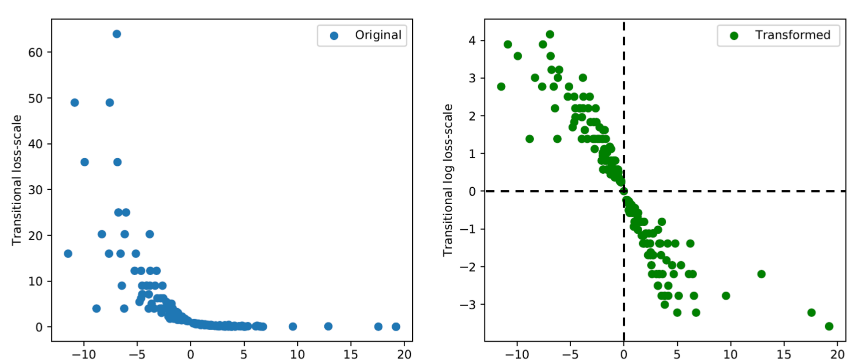

Figure 5: The cross plot between transitional loss-scale and transitional utility-scale. Left is the normal coordinate, while the right is the log coordinate for transitional loss-scale. Some dots in the transformed plot have the same x-axis values with different y-axis values

The result shows the transitional utility-scale r_T and transitional log loss-scale $ln\lambda_T$ have an approximately negative trend and the relationship becomes weak with the increase of $|r|$. It indicates the larger utility-scale has a higher probability of smaller transitional loss-scale.

All the transformed points are distributed in the second and fourth quadrants. The utility-scales in the second quadrant are less than 0, and the log transitional loss-scales are positive, it indicates the risk-seeking in the exponential utility function is corresponding to the left-asymmetric quadratic loss function. The fourth quadrant represents for the risk-averse, it is related to the right-asymmetric quadratic loss function with more penalty on the overestimate.

The boundary for the transitional utility-scale r_T under a given transitional log loss-scale $ln\lambda_T$ caused by the limitation of the interval. In order to better understand the result, the transitional scales could be expressed in Equation 13.

$$\lambda_T = (\frac{a-c}{d-b})^2 \text{ and } \frac{e^{-br_T} - e^{-ar_T}}{e^{-dr_T} - e^{-cr_T}} = \frac{b-a}{d-c} \tag{13}$$

Equation 13 indicates the λ_T is the explicit while the r_T is implicit, which causes the difficulty of the quantitative connection between them. Two special situations could be utilized to explain the dots have the same y-axis values for different x-axis value in Figure 5: ① The experiment one $[c, a, b, d]$ and experiment two $[ck, ak, bk, dk]$, where k is a constant. In this circumstance: λ_T keeps the same, while r_k in experiment two becomes $\frac{1}{k}$ times of itself in experiment one. ② The experiments are changed with the relationship like $a - c = k(d - b)$, where k is a constant. In this circumstance, The r_T is changed while λ_T remains the same.

Discussion

It is evident that loss function is in the opposite of utility function under the same context, for example, the minimum error of the parameter estimate would always bring maximum satisfaction. However, the utility function is widely used in economics, while the loss function is extensively employed in the parameter estimation. The monetary utility function has the characteristics of monotonic increasing and the diminishing marginal utility, which reflects most people own more money would find greater pleasure, and the slope of the monetary utility function would decrease with the increase of money. In the parameter inference, the loss function usually has a shape like a parabola and $\{L(z - z^\star) > 0; z^\star \neq z\}$, which implies both the overestimation and underestimation are out of favor.

In order to investigate a more appropriate form for the loss function, a synthetic one in Equation 14 is considered (Figure 6). It has a similar form with exponential utility, the underestimation is corresponding to the profit and overestimation is related to the loss.

$$L(z - z^\star) = \begin{cases} |(1 - e^{-\lambda(z-z^\star)})/\lambda| & \lambda \neq 0 \\ |x| & \lambda = 0 \end{cases} \tag{14}$$

Where λ is the loss-scale, which is the only parameter that affects the symmetry of the loss function, rather than the weights on two sides in the quadratic loss function. Figure 6 shows the quadratic loss function is similar to the synthetic function, both of them show the same trend but not exactly the same. The optimal estimate of the decision with uniform distribution with an interval between a and b is expressed as:

$$z^\star = -\frac{1}{\lambda} ln \frac{e^{-a\lambda} + e^{-b\lambda}}{2} \tag{15}$$

The relationship between the transitional loss-scale and transitional utility-scale is established with 500 groups of experiments in Figure 7. The intervals are randomly generated in the same way with the exploration of quadratic loss function (Figure 5). The utility-scale around the original has an approximately linear relationship with the loss-scale. In addition. $\lambda > 0$ in the synthetic loss function is corresponding to the risk-averse in the exponential utility function, and the $\lambda < 0$ in this loss function matches with the risk-seeking position in the utility function.

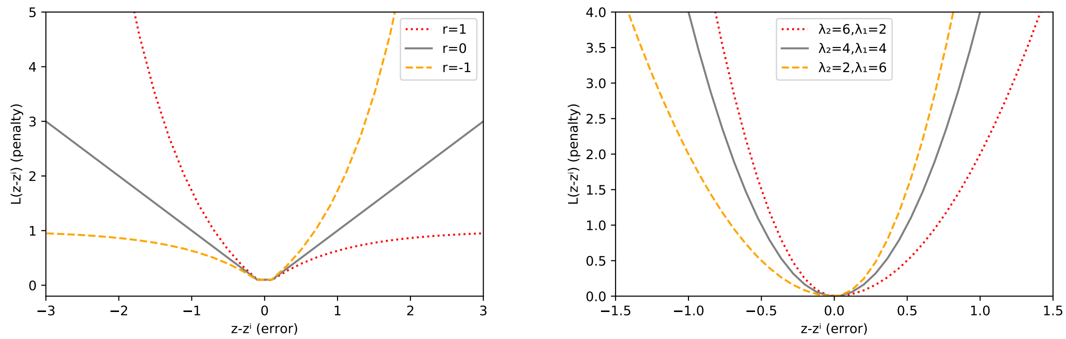

Figure 6: Comparison between the synthetic loss function (left) and quadratic loss function (right)

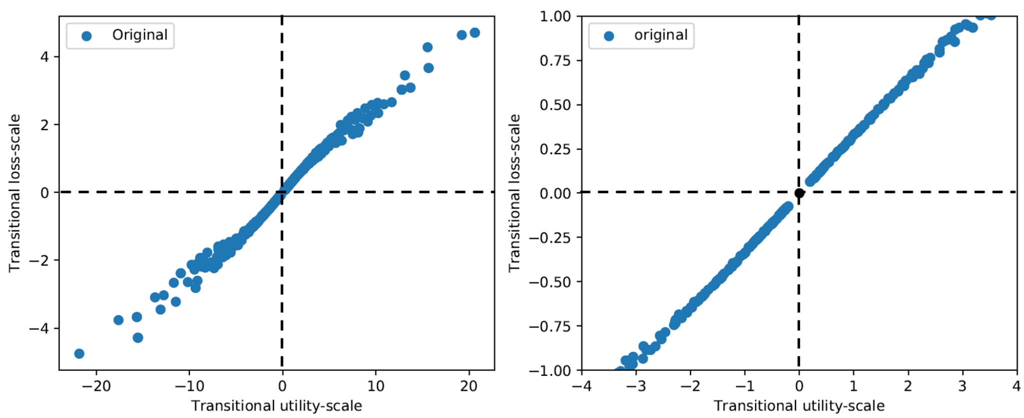

Figure 7: The cross plot between transitional loss-scale and transitional utility-scale in the synthetic loss function. right figure is an enlargement of the left one.

Conclusion

Three basic relationships are established between the risk position in loss function and symmetry of the loss function. ① The symmetric quadratic loss function is equivalent to the exponential utility function in the risk-neutral position, the decision in both functions are based on the expected return. ② The left-asymmetric quadratic loss function with more penalty on underestimated is corresponding to the risk-seeking in exponential utility function ③ The right-asymmetric quadratic loss function with more penalty on overestimation is corresponding to the risk-averse in the exponential utility function.

In addition, the transitional loss-scale has a negative trend with the transitional utility-scale, the smaller loss-scale has a high probability of connecting with the greater utility-scale. Although this relationship is not strict negative, a framework of integration the loss function and utility function in different backgrounds is established.

Finally, a synthetic loss function is proposed in this note, and the loss-scale in this function has an approximately linear relationship with the utility-scale in the exponential utility function around the original. In short, this note constructed a workflow to integrate the utility function and loss function in different backgrounds.

Appendix 1

As for the symmetric quadratic loss function ($\lambda_1 = \lambda_2$):

$$\begin{aligned}\frac{dE\{L\}}{dz^\star} =& \frac{dE\{(z^\star)^2 - 2zz^\star + z^2\}}{dz^\star} \\ =& -2E\{z\} + 2z^\star \\ =& -m + z^\star\end{aligned}$$

Let $\frac{dE\{L\}}{dz^\star} = 0$, the optimal estimate $z^\star = m.$ The expected utility for risk-neutral position is:

$$E(U(x)) = \int xf(x)dx = m \quad \text{For} \quad r = 0$$

Appendix 2

$$\begin{aligned}E\{L(z - z^\star)\} =& \int_a^{z^\star} \lambda_1(z - z^\star)^2 \frac{1}{b-a} dz + \int_{z^\star}^{b} \lambda_2(z - z^\star)^2 \frac{1}{b-a} dz \\ =& \int_a^{z^\star} \lambda_1(z^2 - 2z^\star z + (z^\star)^2) \frac{1}{b-a} dz + \int_{z^\star}^{b} \lambda_2(z^2 - 2z^\star z + (z^\star)^2) \frac{1}{b-a} dz \\ =& \frac{\lambda_1}{b-a}(\frac{z^3}{3}|_a^{z^\star} - 2z^\star \frac{z^2}{2}|_a^{z^\star} + (z^\star)^2(z^\star - a)) + \frac{\lambda_2}{b-a}(\frac{z^3}{3}|_{z^\star}^{b} - 2z^\star \frac{z^2}{2}|_{z^\star}^{b} + (z^\star)^2(b - z^\star)) \\ =& \frac{\lambda_1}{b-a}(\frac{(z^\star)^3}{3} - \frac{a^3}{3} - (z^\star)^3 + z^\star a^2 + (z^\star)^3 - (z^\star)^2 a) \\ &+ \frac{\lambda_2}{b-a}(\frac{b^3}{3} - \frac{(z^\star)^3}{3} - z^\star b^2 + (z^\star)^3 + (z^\star)^2 b - (z^\star)^3) \\ =& \frac{\lambda_1}{b-a}(\frac{(z^\star)^3}{3} - \frac{a^3}{3} + z^\star a^2 - (z^\star)^2 a) + \frac{\lambda_2}{b-a}(\frac{b^3}{3} - \frac{(z^\star)^3}{3} - z^\star b^2 + (z^\star)^2 b))\end{aligned}$$

Taking the derivative wrt $z^\star$, and let $\frac{dE\{L(z-z^\star)\}}{dz^\star} = 0$

$$\begin{aligned}\frac{dE\{L(z - z^\star)\}}{dz^\star} =& \frac{\lambda_1}{b-a}((z^\star)^2 + a^2 - 2az^\star) + \frac{\lambda_2}{b-a}(-(z^\star)^2 - b^2 + 2bz^\star) = 0 \\ =& \lambda_1((z^\star)^2 + a^2 - 2az^\star) + \lambda_2(-(z^\star)^2 - b^2 + 2bz^\star) = 0 \\ =& (\lambda_1 - \lambda_2)(z^\star)^2 + 2(b\lambda_2 - a\lambda_1)z^\star + \lambda_1 a^2 - \lambda_2 b^2 = 0\end{aligned}$$

$$z^\star = \frac{-2(b\lambda_2 - a\lambda_1) \pm \sqrt{4(b\lambda_2 - a\lambda_1)^2 - 4(\lambda_1 - \lambda_2)(\lambda_1 a^2 - \lambda_2 b^2)}}{2(\lambda_1 - \lambda_2)}$$

$$z^\star = \frac{-(b\lambda_2 - a\lambda_1) \pm \sqrt{\lambda_1\lambda_2}|a - b|}{(\lambda_1 - \lambda_2)} = \frac{-(b\lambda - a) \pm \sqrt{\lambda}(b - a)}{(1 - \lambda)} \quad \text{Where} \quad \lambda = \frac{\lambda_2}{\lambda_1}$$

$$z^\star = \frac{a + \sqrt{\lambda} b}{1 + \sqrt{\lambda}} \quad \text{or} \quad \frac{a - \sqrt{\lambda} b}{1 - \sqrt{\lambda}}$$

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